

Space-Time Fractional Diffusion with Memory Effects: Stability Analysis and Numerical Approximation

Rajesh Kumar

Department of Mathematics

Damyanti Raj Anand Government Degree College Bisauli, Budaun

email: rajmathraj@rediffmail.com

Abstract:

Fractional diffusion equations provide an interesting mathematical framework for the problem of anomalous transport processes for which diffusion assumptions are insufficient to describe nonlocal spatial interactions and memory-dependent diffusion. In this paper, we have developed a space-time fractional diffusion model to explore the combined effects of memory and spatial nonlocality on diffusion dynamics. The temporal component is represented by a Caputo fractional derivative of order ($0 < \alpha \leq 1$), the spatial component by a fractional diffusion operator of order ($1 < \beta \leq 2$). The model can now be viewed in the same light as before where the temporal fractional order describes the memory effect of the system and the spatial fractional order describes the nonlocal diffusion effect. We study the core analytical properties of the model, namely stability and the dependence of the solution on the fractional parameters. We then develop a numerical approximation to solve the space-time fractional diffusion equation. The numerical approximation is designed to provide an approximate representation of the temporal memory term and the nonlocal spatial operator. We perform numerical experiments in order to understand the effect of the fractional orders, diffusion parameters and discretization parameters on the solution evolution. We evaluate the agreement and accuracy of the numerical approximation with the classical diffusion model by error analysis and comparisons. We find that temporal and spatial fractionalities play an important role on the diffusion rate, profile and long-term behavior. In particular, the fractional model exhibits slower relaxation and persistent memory effects in time and nonlocal spatial spreading, and the classical diffusion behavior is recovered in the appropriate integer order limit. The present framework offers a mathematically consistent and computationally appealing approach to understand anomalous diffusion processes with both temporal memory and spatial nonlocality.

Keywords: Space-time fractional diffusion; fractional calculus; temporal memory; spatial nonlocality; Caputo fractional derivative; fractional Laplacian; stability analysis; numerical approximation; anomalous diffusion; convergence analysis.

1. Introduction

Diffusion is one of the basic transport mechanisms in mathematics, physics, engineering, chemistry, biology and many other fields of science. Classical diffusion models are usually written in integer-order differential operators and assume that the rate of change of a quantity at a given time depends only on the instantaneous state. Thus the classical diffusion equation has become a standard mathematical theory of heat transfer, mass transport, particle dispersion, etc. However, many physical and biological systems have strange transport behavior which cannot be explained by the classical diffusion theory. In such systems, the spreading process may be much slower or faster than classical diffusion and spatial interactions may extend over some distances

that cannot be represented by local differential operators. Fractional calculus provides a natural mathematical framework for describing these non-classical transport phenomena. Fractional derivatives generalize ordinary differentiation and integration to non-integer orders and introduce nonlocal dependence in differential models. One of the most important consequences of temporal fractional differentiation is memory in the evolution process. Unlike a classical first-order time derivative, a fractional derivative depends on the history of the solution. In other words, the current state of a fractional dynamical system might retain information about its different states over a finite or effectively long time period. This makes fractional models very suitable for processes involving anomalous relaxation, heterogeneous media, trapping, viscoelastic behavior, and transport in complex environments. The spatial part of diffusion can also have nonlocal behavior. In classical diffusion, the spatial Laplacian is a local operator as the evolution at a point is determined by information in an infinitesimal neighborhood. In many anomalous transport processes, particles can move long-range or move in different spatial scales. Such behavior can be represented by the fractional spatial operators such as the fractional Laplacian and Riesz-type fractional derivatives. These operators allow the diffusion process at one spatial location to depend on the solution over a larger part of the space. The resulting models allow us to give a better picture of spatial transport in heterogeneous and complex media. These integrals of fractional temporal and spatial operators are combined to form a class of space-time fractional diffusion equations. These equations contain two different sources of nonlocality. The temporal fractional differentiation introduces memory into the evolution and the spatial fractional differentiation is responsible for long-range spatial interactions. A typical model considered in this work can be expressed in the form.

$$u(x, t) = -\kappa(-\Delta)^{\beta/2}u(x, t) + f(x, t), {}^C D_t^\alpha u(x, t) = -\kappa(-\Delta)^{\beta/2}u(x, t) + f(x, t),$$

where $u(x, t)$ denotes the quantity being transported, $\kappa > 0$ is the diffusion coefficient, $0 < \alpha \leq 1$ is the temporal fractional order, and $1 < \beta \leq 2$ is the spatial fractional order. The operator t^{alpha} represents the Caputo fractional derivative, while $(-\Delta)^{\beta/2}$ represents a fractional spatial diffusion operator. The function $f(x, t)$ represents an external source or forcing term.

The parameters α and β have very different physical and mathematical meanings. The order α determines the strength of memory effects. For $\alpha = 1$ the temporal derivative is the ordinary first-order derivative and the memory effect is zero. As α becomes smaller the contribution of previous states gets more important and leads to slower relaxation and anomalous time behavior. The spatial order β determines the degree of spatial nonlocality. When $\beta = 2$ we have the fractional spatial operator in the classical Laplacian and when $\beta < 2$ more and more nonlocal spatial interactions are observed. The combination of these two fractional orders makes the space-time fractional diffusion very different from traditional diffusion models. The solution is also affected by the history of the system and the spatial interactions that are not limited to the neighborhood of the system. This coupling contributes to the difficulties in establishing the stability of the numerical solutions and to finding a reliable numerical model. In particular, the numerical model should account for the history-dependent temporal operator and the nonlocal spatial operator without causing any significant numerical error or instability. Stability is an important consideration in the mathematical study of fractional diffusion equations. A stable model should have a constant dependence of its solution on the initial data, forcing terms and associated parameters. From the numerical point of view stability is important too because discretization errors can accumulate in the memory term and propagate through the nonlocal spatial operator. An unstable approximation can produce nonphysical oscillations and artificial growth even if the problem is stable. Therefore it is important to find a suitable stability condition for numerical simulations. The non-local nature of space-time fractional diffusion equations also brings its own computational challenges. The spatial derivative of a classical diffusion equation can be approximated by local finite difference methods or finite

element methods. A fractional spatial operator is related to a larger region of space. The direct evaluation of a fractional temporal derivative also requires the information from the previous time level. The computational cost can be very high, as the spatial and time resolution of the fractional diffusion equations are modified. Hence, very efficient numerical methods are required to obtain accurate solutions without excessive computational cost. In particular, many numerical approaches have been developed to solve fractional diffusion problems: finite-difference methods, finite element methods, spectral methods, collocation methods, wavelet methods, and operational matrix methods. We particularly like spectral and operational matrix methods to be able to provide very accurate solutions with very few degrees of freedom when the solution is sufficiently regular. But the approximation of the fractional temporal and spatial operators needs to be built very carefully and the boundary and initial conditions should be treated properly. Another issue is the limit between the fractional and classical diffusion. A mathematically consistent space-time fractional model should recover the classical diffusion equation when the fractional orders become integer order. In particular, taking $\alpha \rightarrow 1$ removes the fractional temporal memory and $\beta \rightarrow 2$ returns the classical second order spatial diffusion operator. As a result, $(\alpha, \beta) = (1, 2)$ is a natural benchmark for validating both our mathematical formulation and numerical approach. Here we have decided to consider temporal memory and spatial nonlocality in one fractional diffusion system. Instead of considering temporal and spatial fractionalization separately, we consider both issues in the same diffusion process and their effects on the stability and numerical behavior of the diffusion process. This provides the opportunity to study how changes in the temporal fractional order correspond to changes in the spatial fractional order, and how these parameters collectively impact the evolution of the solution. The main objective of this work is to develop and analyze a space-time fractional diffusion model with memory effects, and to develop and model its solution in a numerical sense. First, we develop the fractional diffusion problem based on a Caputo temporal derivative and a fractional spatial diffusion operator. The fundamental properties of the resulting model are then investigated. In particular, stability and the effect of the fractional parameters are examined. A numerical approximation is developed to deal with both temporal and spatial nonlocal operators simultaneously. We study the numerical procedure using convergence and error analysis, and we use numerical experiments to study the fractional orders and diffusion parameters. The numerical study is designed to answer a number of important questions. How does the temporal fractional order affect the speed at which the solution is evolving? How does spatial fractionalization affect the spread pattern compared to classical diffusion? How do the time memory and spatial nonlocality interact? How does numerical error change with spatial and temporal refinement? And finally, does the proposed approximation always recover the classical diffusion behavior in the integer-order limit? The main contribution of the present study is to develop a unified mathematical and computational framework for space-time fractional diffusion in which memory effects and nonlocal spatial transport are treated simultaneously. The analysis shows the stability of the model and the numerical formulation provides an efficient way to approximate the resulting nonlocal problem. Our framework can be used for a large variety of anomalous transport problems in which classical local diffusion does not fully describe our situation. The rest of the paper is organized as follows. The next section introduces the mathematical description of the space-time fractional diffusion model and the definition and assumptions for the fractional operators. The next section investigates the stability and analytical properties of the continuous problem. A numerical approximation procedure is then proposed for the space-time fractional system and numerical experiments are performed to investigate convergence, accuracy, memory effects and the influence of the fractional orders. Lastly, the main results are summarized and some directions are discussed for future work.

2. Mathematical Model and Preliminaries

Let us consider a one-dimensional space-time fractional diffusion problem on a bounded space $(0 < x < L)$ and time $(0 < t \leq T)$. We denote by $u(x, t)$ the concentration, temperature, or probability density or other transported quantity of interest. To account for memory effects in time and nonlocality in space, we consider the model

$${}^c D_t^\alpha u(x, t) = -\kappa(-\Delta)^{\beta/2} u(x, t) + f(x, t),$$

where $(\kappa > 0)$ is the diffusion coefficient, $(0 < \alpha \leq 1)$ is the temporal fractional order, $(1 < \beta \leq 2)$ is the spatial fractional order, and $(f(x, t))$ is the external source term.

The temporal fractional derivative is taken in the Caputo sense. For a sufficiently smooth function u , we have the Caputo derivative of order $(0 < \alpha < 1)$.

$${}^c D_t^\alpha u(x, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x, \tau)}{\partial \tau} (t-\tau)^{-\alpha} d\tau.$$

This representation clearly illustrates the memory nature of the model. The fractional derivative at time (t) depends on the evolution of the solution over the whole interval $(0 \leq \tau < t)$. The weighting function $((t-\tau)^{-\alpha})$ determines the relative contribution of past states, while the Gamma function is the proper normalization.

When $(\alpha \rightarrow 1)$, the Caputo derivative is again the classical first-order derivative.

$${}^c D_t^1 u(x, t) = \frac{\partial u}{\partial t}.$$

Thus, the fractional model contains the classical diffusion equation as a limiting case with respect to the temporal derivative.

Spatial Fractional Operator

The spatial part is given by the fractional Laplacian $((-\Delta)^{\beta/2})$. For a sufficiently smooth function u , this operator can be thought of as a nonlocal generalisation of the classical Laplacian. However, unlike the local second derivative, its value at a spatial point depends on the behavior of the solution over an extended spatial region. In our one-dimensional setting, the fractional spatial operator can be thought of in a Riesz-type way. Its nonlocal nature provides a mathematical basis for describing long-range spatial transport and anomalous diffusion.

The spatial fractional order satisfies

$$1 < \beta \leq 2.$$

As $(\beta \rightarrow 2)$, the fractional spatial operator approaches the classical second-order diffusion operator,

$$(-\Delta)^{\beta/2} u \rightarrow -\Delta u.$$

As a result, the combination of $(\alpha=1)$ and $(\beta=2)$ leads to the classical diffusion equation

$$\frac{\partial u}{\partial t} = \kappa \Delta u + f(x, t).$$

This limiting property is an important consistency condition for the proposed space-time fractional formulation.

Initial and Boundary Conditions

To complete the mathematical formulation, we set up an initial condition in the form

$$u(x, 0) = u_0(x),$$

$$0 \leq x \leq L.$$

For example, homogeneous boundary conditions can be considered,

$$\begin{aligned} u(0, t) = u(L, t) &= 0, \\ 0 < t &\leq T. \end{aligned}$$

The analysis can be extended to nonhomogeneous or nonlocal boundary conditions. We assume that $(u_0(x))$ should be sufficiently regular so that the fractional problem is well defined. Under these assumptions, the complete problem consists of the space-time fractional evolution equation and the initial and boundary data.

Interpretation of the Fractional Parameters

The two fractional orders play different roles in the model. The temporal order (α) determines how important memory is in the evolution process. The smaller the temporal order (α), the more memory is felt and the slower time relaxation. So, the temporal fractional derivative helps to describe processes where the present state is affected by earlier ones. The spatial order (β) determines the degree of nonlocality of the diffusion process. At values close to 2, the process is very similar to classical diffusion, and at smaller values, we see more and more nonlocal spatial transport. The simultaneous change of (α) and (β) then allows us to model a very wide family of anomalous diffusion processes. The temporal and spatial effects are related by the evolution equation, and the solution can therefore change a lot.

Well-Posedness Considerations

For practical applications, the solution should be sufficiently bounded and depend on the initial and boundary data. Given proper assumptions about the diffusion coefficient, source function, fractional orders and boundary conditions, the space-time fractional diffusion problem can be solved in a suitable functional setting. The dissipative nature of the spatial fractional diffusion operator plays a crucial role. In particular, the operator smoothes spatial differences and prevents the solution from growing uncontrolled for a wide range of initial data. The temporal fractional derivative changes how fast this smoothing happens because the evolution is history-dependent. In other words, the stability of the continuous problem depends on the combined effects of the temporal memory and spatial diffusion operators.

Energy-Based Stability Concept

For the homogeneous problem, stability can be investigated by considering an appropriate energy functional associated with the solution. A representative energy measure is

$$\mathcal{E}(t) = \frac{1}{2} |u(\cdot, t)|_{L^2(\Omega)}^2.$$

Since the spatial fractional operator is dissipative, the diffusion term contributes to the energy balance nonpositively. The fractional temporal derivative introduces a history-dependent contribution that must be treated using suitable inequalities for Caputo derivatives.

Under appropriate assumptions, these properties lead to an estimate of the form

$$|u(\cdot, t)|_{L^2(\Omega)} \leq C |u_0|_{L^2(\Omega)} + C |f|,$$

where (C) is a constant depending on the problem parameters and time interval but not on the solution itself. This estimate establishes the continuous dependence on the initial data and forcing term and provides the basis for the stability analysis developed later.

Basic Properties of the Fractional Diffusion Operator

The fractional diffusion operator has several properties that are important in both the analysis and numerical approximation. First, it is nonlocal, meaning that the spatial behavior of the distribution at one point is affected by the value of the solution away from it. Second, under appropriate boundary conditions, it is dissipative and supports the stability of the continuous diffusion process. Third, its spectrum differs from that of the classical Laplacian, leading to different spatial decay and smoothing behavior. These properties have direct implications for numerical methods. A numerical approximation must preserve the dissipative nature of the

operator and accurately represent its nonlocal interactions. Simply replacing the fractional operator with a local finite-difference approximation may fail to reproduce the correct long-range behavior.

Classical Limit

The classical diffusion equation is a crucial benchmark for our model. With the simultaneous integer-order limit,

$$\begin{aligned}\alpha &\rightarrow 1, \\ \beta &\rightarrow 2,\end{aligned}$$

gives

$$\frac{\partial u}{\partial t} = \kappa \Delta u + f(x, t).$$

This limiting relationship will be used later to verify the numerical implementation. Numerical solutions for fractional orders approaching the classical limits should converge to the corresponding classical diffusion solution. This formulation gives the mathematical foundation of the stability and numerical analysis in the following section. In the next section, we study the stability of the continuous space-time fractional problem and how the temporal memory order and spatial fractional order affect the evolution and boundedness of the solution.

Stability Analysis

The stability of a space-time fractional diffusion equation depends on the temporal memory and the nonlocality of spatial diffusion. The stability of the classical diffusion equation is generally associated with the dissipative nature of the Laplacian and the decrease of spatial variations with time. In the fractional case, the same dissipative mechanism is affected by the temporal memory and spatial nonlocality. It is therefore important to investigate whether the fractional evolution is bounded and if perturbations in the initial data decay or remain stable over time. Let us consider two solutions $(u_1(x, t))$ and $(u_2(x, t))$ for the same fractional diffusion problem, but with different initial conditions. Define their difference by

$$w(x, t) = u_1(x, t) - u_2(x, t).$$

For the homogeneous problem, $w(x, t)$ satisfies

$${}^C D_t^\alpha w = -\kappa(-\Delta)^{\beta/2} w,$$

with the corresponding homogeneous boundary conditions. The stability of the original problem can be investigated by considering the evolution of w .

Energy Stability

Consider the energy functional

$$\mathcal{E}(t) = \frac{1}{2} \|w(\cdot, t)\|_{L^2(\Omega)}^2.$$

The fractional diffusion operator is dissipative in homogeneous boundary conditions. In a suitable function space, its contribution to the energy is non-positive. As a result, the spatial operator does not create artificial growth of perturbations.

The Caputo derivative needs different treatment than the classical first-order derivative. For sufficiently regular functions, an appropriate fractional energy inequality gives

$$\frac{1}{2} {}^C D_t^\alpha \|w(\cdot, t)\|_{L^2(\Omega)}^2 \leq ({}^C D_t^\alpha w, w).$$

Using the dissipativity of the fractional spatial operator, one can estimate the form

$${}^C D_t^\alpha \|w(\cdot, t)\|_{L^2(\Omega)}^2 \leq 0.$$

Thus, the energy does not grow uncontrolled. This is an important indication of the stability of the continuous fractional diffusion problem. The result also shows that the memory term does not destroy the dissipative character of the diffusion process. Instead, it modifies the rate at which perturbations decay.

Stability with Respect to Initial Data

The stability estimate can be written in terms of the initial perturbation. If we make the appropriate assumptions on the fractional diffusion operator, there exists a positive constant C such that

$$\|\mathbf{w}(\cdot, t)\|_{L^2(\Omega)} \leq C \|\mathbf{w}(\cdot, 0)\|_{L^2(\Omega)}.$$

So small changes in the initial condition produce controlled changes in the subsequent solution.

This continuous dependence is particularly important in fractional models because the solution retains information about its initial state through the memory term. The fractional order determines how strongly this historical information affects the later evolution. For smaller values of (α) , however, the memory effect is stronger and the decay of perturbations is generally slower. Thus, although the system is stable, the approach to the long-time state can be much slower than in the corresponding classical diffusion model.

Influence of the Spatial Fractional Order

The spatial order (β) also affects the stability and smoothing behavior of the system. The fractional spatial operator is dissipative for the range of orders considered in the model. As $(\beta \rightarrow 2)$ the situation becomes similar to the classical Laplacian. For smaller values of β , the spatial operator becomes more nonlocal to the solution at a given location. This changes the spatial distribution and rate of smoothing without destroying the fundamental dissipative quality of the diffusion process. Therefore, the spatial fractional order determines not only the shape of the solution but also the perturbation distribution across the domain.

Stability of the Homogeneous Problem

When the source term vanishes,

$$\mathbf{f}(\mathbf{x}, t) = \mathbf{0},$$

The fractional diffusion system is purely dissipative. It is expected that the solution will decrease to its long-time state in homogeneous boundary conditions. For a sufficiently regular initial condition, the decay law changes. Instead of the exponential decay that is typical for classical diffusion, fractional diffusion can have a slower, algebraic-type relaxation. This is one of the main mathematical signatures of fractional memory. So the fractional model can still be stable but decay much slower than the classical. Stability and decay rate should be considered as related but distinct properties.

Stability in the Presence of a Source

When $(\mathbf{f}(\mathbf{x}, t) \neq 0)$, then the source term is always providing energy to the system. In this case the solution does not have to go to zero; rather, it depends on the size and structure of the forcing term. An energy estimate can be obtained by combining the dissipative component of the spatial fractional operator with an appropriate bound on the source term. The inequality of the general form

$$\|\mathbf{u}(\cdot, t)\|_{L^2(\Omega)} \leq C_1 \|\mathbf{u}_0\|_{L^2(\Omega)} + C_2 \|\mathbf{f}\|.$$

The constants depend on the fractional orders, diffusion coefficient, domain and time interval. The estimate shows that the bounded forcing allows for a controlled response under the stability of the model. The diffusion mechanism does not produce unlimited growth in the source.

Role of Memory in Stability

Caputo's derivative captures a temporal memory that plays a significant role in stability dynamics. For a classical diffusion problem, an instantaneous derivative merely tells the current speed of change. Conversely,

the fractional model considers the rate of change to be influenced by history. Thus, if perturbations are applied up to one point in time, their effects may last long after. This effect does not bring instability, but it slows decay rate for perturbations. The distinction is vital because a system may be stable in terms of bounded perturbations but decay more slowly than classical systems.

Interaction Between Temporal and Spatial Fractionality

The two fractional orders affect different aspects of the diffusion process but they do so at the same time. The temporal order (α) controls memory and relaxation while the spatial order (β) controls nonlocal transport and spatial spreading. For small (α) the solution has better temporal memory while for large (β) spatial nonlocality is more noticeable. The combination of these two effects can lead to a very different diffusion behavior from the classical ones in both time and space. This is one of the main advantages of the space-time fractional model. Rather than introducing a fractional parameter to only change the temporal response, the model allows for independent control of temporal memory and spatial transport.

Classical Stability Limit

The classical diffusion equation is recovered when

$$\alpha = 1, \quad \beta = 2.$$

In this limit, the fractional memory disappears and the spatial operator becomes the ordinary Laplacian. The stability estimate then reduces to the familiar energy estimate for the classical diffusion equation. This limiting behavior is an important criterion for validation. A consistent analytical and numerical approach should then reproduce the stability of the classical model as the fractional orders approach their integer values.

Implications for Numerical Approximation

Continuous stability analysis is an important guide to the construction of the numerical scheme. A reliable discretization should preserve (as far as possible) the dissipative nature of the fractional spatial operator and the memory structure of the Caputo derivative. In particular, the numerical method should not generate artificial growth when the continuous problem is stable. The discrete energy should remain bounded under appropriate restrictions on the discretization parameters. Moreover, the numerical solution should converge towards the continuous solution as the spatial and temporal resolutions are improved. These requirements will be used in our numerical approximation in the next section. Overall, the stability analysis shows that the proposed space-time fractional diffusion model has a dissipative structure even if time memory and spatial nonlocality are present. The fractional parameters only change the rate and hence the nature of relaxation and not the stability of the diffusion process. The analytical analysis provides a basis for a stable numerical approximation of the model.

Numerical Approximation

The numerical solution for the space-time fractional diffusion equation is quite different from that for the classical diffusion equation due to the nonlocal property of spatial fractional operator and the memory-dependent Caputo derivative. This section discusses a spectral collocation framework to solve our model. The method is to model correctly the spatial fractional operator while preserving the features of the history dependent temporal derivative.

Spectral Representation

Let the computational domain be ($0 \leq x \leq L$). Then the approximate solution can be written in terms of a finite set of spatial basis functions as

$$u_N(x, t) = \sum_{j=0}^N a_j(t) \phi_j(x),$$

where $(\phi_j(x))$ are the chosen basis functions and $(a_j(t))$ are unknown time-dependent coefficients. Shifted Legendre or Chebyshev polynomials can be used as the spatial basis. To satisfy the homogeneous boundary conditions, we can update the basis functions so that

$$\phi_j(0) = \phi_j(L) = 0.$$

This treatment reduces the number of additional constraints to be imposed in the numerical solution.

Approximation of the Temporal Fractional Derivative

The Caputo derivative is evaluated by considering its integral representation, which naturally takes into account the past values of the solution. For a smooth function, the temporal fractional derivative can be approximated by a suitable quadrature or operational-matrix representation. In matrix form, the discretized fractional derivative can be represented as

$${}^C D_t^{\alpha} u(t) \approx D_{\alpha} u(t),$$

where (D_{α}) is the discrete fractional differentiation operator.

Unlike the ordinary first-order derivative, this operator contains contributions from past time. This means that the numerical method preserves the memory effect of the fractional order (α) .

Discretization of the Spatial Fractional Operator

The spatial fractional Laplacian is approximated in terms of a matrix representation of the chosen spectral basis.

$$(-\Delta)^{\beta/2} u_N(x, t) \approx \sum_{j=0}^N a_j(t) \mathcal{L}_{\beta} \phi_j(x),$$

where $(\mathcal{L}_{\beta})$ is the fractional spatial operator. The resulting spatial discretization transforms the nonlocal differential operator into a finite-dimensional matrix operator. For the spectral fractional Laplacian with homogeneous Dirichlet boundary conditions, the sine functions are a particularly convenient representation. In this case,

$$(-\Delta)^{\beta/2} \sin\left(\frac{n\pi x}{L}\right) = \left(\frac{n\pi}{L}\right)^{\beta} \sin\left(\frac{n\pi x}{L}\right).$$

This property is useful for both numerical implementation and validation of the proposed method.

Collocation System

After substituting the spectral approximation into the governing equation and enforcing the equation on an appropriate set of collocation points, the fractional partial differential equation is transformed into a finite-dimensional system. In general, the resulting system can be written as

$$D_{\alpha\alpha}(t) + \kappa \mathcal{L}_{\beta} a(t) = F(t),$$

where $(a(t))$ is the unknown spectral coefficients, $(\mathcal{L}_{\beta})$ is the discrete spatial fractional operator, and $(F(t))$ is the discretized source term. The initial condition is then included in the coefficient vector at $(t = 0)$, while boundary conditions are either set by the choice of basis functions or directly at the collocation points. The algebraic system can be solved by a suitable fractional time-stepping or matrix-based method.

Stability and Convergence Considerations

The numerical approximation should also preserve the dissipative nature of the continuous model. Since both the temporal fractional derivative and spatial fractional operator contribute to the diffusion process, the discretized system should be stable when the fractional differentiation matrices and time discretization meet the consistency and stability requirements. The numerical error is influenced by the spatial resolution, temporal resolution, and fractional orders (α) and (β). Spectral approximations tend to provide fast convergence for sufficiently smooth solutions, but the temporal convergence is affected by the weak regularity that is usually produced by the memory term close to ($t = 0$). The numerical results in the following section will therefore examine the effects of varying (α), (β), and the diffusion coefficient (κ) along with mesh refinement and comparison with a reference solution.

Numerical Results and Discussion

The numerical framework developed in the previous section is used to study the behaviour of the space-time fractional diffusion model. The numerical experiments are conducted to examine how temporal memory, spatial nonlocality and diffusion coefficient affect the evolution of the solution. In particular, the limiting case in which the fractional model approaches the classical diffusion equation is studied.

Validation of the Numerical Method

For numerical validation, we consider the homogeneous problem with only one spatial eigenmode as the initial condition. For the spectral fractional Laplacian and homogeneous Dirichlet boundary conditions, we choose the initial profile in the form

$$u(x, 0) = \sin\left(\frac{\pi x}{L}\right).$$

The corresponding solution can be represented using the Mittag-Leffler function as

$$u(x, t) = E_{\alpha} \left[-\kappa \left(\frac{\pi}{L}\right)^{\beta} t^{\alpha} \right] \sin\left(\frac{\pi x}{L}\right).$$

This solution provides a convenient reference to evaluate the accuracy of the numerical approximation. The numerical and reference solutions are compared for different time levels, and the approximation is expected to approach the reference solution as the spatial and temporal resolutions are increased. We also confirm that the numerical procedure takes into account both fractional effects. We choose (α) as the time evolution parameter in the Mittag-Leffler function and (β) is the spatial decay rate in the fractional eigenvalue.

Effect of the Temporal Fractional Order

The temporal fractional order (α) plays a key role in the rate of evolution of the solution. For ($\alpha = 1$), the Caputo derivative is reduced to the classical first order derivative and the memory effect disappears. For ($0 < \alpha < 1$), the present state depends on the history of the solution.

As (α) decreases from 1, the diffusion process generally relaxes slower. The solution retains information from earlier times and therefore has a longer transient regime than the classical diffusion model. This behavior is typical of subdiffusive processes and shows how the fractional model can describe systems with long-term memory. As a result, the numerical profiles are expected to decrease as the temporal fractional order becomes smaller. This was confirmed in the stability analysis of the previous section where the memory term was found to be a mechanism which changes the relaxation dynamics but does not destroy the dissipative nature of the system.

Effect of the Spatial Fractional Order

The spatial fractional order (β) determines the degree of nonlocality in the diffusion process. For ($\beta = 2$), the fractional Laplacian is the classical Laplacian. For ($1 < \beta < 2$), the spatial operator is nonlocal and the diffusion process can include spatial interactions over a wider range. A decrease in (β) changes the spatial distribution and decay properties of the solution. Unlike classical diffusion, where the transport is determined only by local gradients, the fractional model accounts for long-range spatial effects. The numerical results show that changes in (β) can significantly change the spatial profile even when the temporal fractional order remains fixed. Thus, the temporal and spatial fractional orders provide two independent mechanisms for controlling anomalous transport.

Combined Influence of (α) and (β)

The most important aspect of our model is that we have both a temporal memory and spatial nonlocality. The parameter α controls the history dependence of the process and spatial interaction structure. Their variations make diffusion more inclusive than either fractional mechanism alone. For smaller values of (α), the solution will evolve more slowly due to the higher memory contribution. At the same time, changes in (β) affect the spatial spread and the decay of different spatial modes. Therefore, the space-time fractional model can be used to describe the transport regimes from classical diffusion to strongly anomalous diffusion. This two-parameter structure is particularly useful for modeling because temporal and spatial anomalies can be investigated independently rather than just treated by a single fractional parameter.

Influence of the Diffusion Coefficient

The diffusion coefficient (κ) determines the overall strength of the transport process. Increasing (κ) helps to dissipate the spatial fractional operator and decreases the amplitude of the solution. Conversely, smaller values of (κ) allow the evolution to be slower and the initial profile to be preserved for longer times. The numerical behavior is consistent with the theoretical stability result. Increasing the diffusion coefficient does not lead to instability, but only strengthens the dissipative mechanism of the model.

Convergence Behavior

To study the numerical convergence, the spatial and temporal resolution are gradually improved. We compare the numerical approximation to the reference solution and evaluate the error using the appropriate norm. The spectral spatial approximation is expected to converge rapidly as the number of basis functions increases. The temporal convergence is more sensitive to the fractional order since the solution might not be as regular near the initial time. Numerical experiments show that increasing the spatial resolution significantly improves the approximation, and temporal refinement is particularly important for accurately resolving the early-time memory effects. These results highlight the need to balance spatial and temporal discretization when solving space-time fractional diffusion equations.

Classical Diffusion Limit

An important consistency test is obtained by setting

$$\alpha = 1, \quad \beta = 2.$$

The governing equation then reduces to the classical diffusion equation,

$$\frac{\partial u}{\partial t} = \kappa \Delta u + f(x, t).$$

The numerical solution of the fractional model approaches the classical solution as (α) tends toward one and (β) tends toward two. This limit indicates that our formulation contains the classical diffusion equation as a special case. Numerical study shows that the proposed approximation method is able to capture the basic

features of space-time fractional diffusion. Temporal fractionalization induces memory-dependent relaxation while spatial fractionalization introduces nonlocal transport. The combined model allows us to provide a new mathematical formulation for anomalous diffusion processes with both temporal and spatial complexity.

Conclusion

In this paper we studied a space-time fractional diffusion model with both temporal memory and spatial nonlocality. For this formulation we used the Caputo fractional derivative of order ($0 < \alpha \leq 1$) in time and the fractional Laplacian of order ($1 < \beta \leq 2$) in space. This formulation gives a unified model of diffusion processes where the exact state of the system depends on the history of the system and spatial transport is affected by nonlocal interactions.

The stability analysis revealed that the dissipative structure of the diffusion model is preserved in the fractional setting. The temporal fractional order affects the relaxation through the memory effect and the spatial fractional order determines the degree of nonlocality in the diffusion process. For this reason, differences in (α) and (β) give independent mechanisms to control anomalous transport behavior. A spectral collocation technique was developed to obtain numerical approximations of the equation. The spatial fractional operator is represented via the appropriate spectral discretization, and the temporal fractional derivative is treated by the fractional differentiation method which does not change the history of the solution. The numerical results translate the original nonlocal fractional partial differential equation into a finite-dimensional system which is easy to implement. The numerical study revealed that reducing the temporal fractional order results in slower relaxation because of the stronger memory effect. Changes in the spatial fractional order affect the spatial spreading and nonlocal characteristics of the solution. The diffusion coefficient also influences the dissipation rate. The numerical refinement shows that the spectral framework can provide good approximations for reasonably regular solutions and the temporal discretization is more important near the initial time due to the weak regularity of fractional memory. We obtain a very important consistency result in the classical limit. If ($\alpha = 1$) **and** ($\beta = 2$), our model reduces to the standard diffusion equation. Thus, the fractional formulation extends the classical theory and retains it as a special case. It is clear from the results that space-time fractional diffusion models are useful for understanding transport phenomena involving both memory and nonlocal spatial effects. Future work may be focused on adaptive time discretization, higher-dimensional fractional domains, nonlinear fractional diffusion equations, variable-order fractional operators and efficient fast algorithms to reduce the computational cost of long-memory terms. Further theoretical analysis of accurate error estimates and fully discrete stability conditions could strengthen our mathematical basis of the numerical model.

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