Bicomplex Sequence Spaces Generated by Multi-Norms

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Abstract:
Classical coordinate methods, idempotent-valued norms, and zero-divisor algebra are combined in bicomplex sequence spaces. The spaces $c_0(\mathscr{B}\mathbb{C})$, $c(\mathscr{B}\mathbb{C})$, $\ell^p(\mathscr{B}\mathbb{C})$, and $\ell^\infty(\mathscr{B}\mathbb{C})$ are introduced in this paper. Subsequently, sequence spaces are generated by a bicomplex multi-norm. A pair of complex sequence spaces is isometrically identified with each classical bicomplex space. The following are derived componentwise: completeness, inclusion relations, density of finitely supported sequences, coordinate projections, shifts, Schauder bases, continuous duals, and Köthe duals. The inequality $\|x\|_{q,\mathbb{D}} \preccurlyeq_{\mathbb{D}} \|x\|_{p,\mathbb{D}}$ is established for counting-measure sequence spaces when $1 \le p \le q \le \infty$. The domain dependence of the inequality is underscored. Using $\sup_n N_n(x_1,\ldots,x_n)$, a prefix-bounded multi-norm space $m_N(E)$ is defined and proven to be Banach whenever E is complete. The minimum multi-norm is demonstrated to reduce to $c_0(E)$ in the analysis of a tail-vanishing subspace. Operator-generated weighted spaces, matrix transformations, and a coordinate-tail criterion for compact operators are established. The unit-vector sequence in ℓ^1 serves as a counterexample, while weak convergence is characterised for reflexive ℓ^p ranges and c_0. Geometric, basis, shift, diagonal, matrix, and multi-norm-generated sequences are among the examples that have been implemented. Open problems pertain to the spectral properties of sequence operators, weak compactness, bicomplex matrix domains, Köthe echelon constructions, and nonminimum multi-norm duality.

Keywords: bicomplex sequence space; ℓp space; multi-norm-generated sequence; Schauder basis; Köthe dual; matrix transformation; compact operator; weak convergence.

Mathematical Notation and Symbols

Symbol	Meaning
$x=(x_n)$	Bicomplex sequence with $x_n=x_{n,1}\mathbf{e}_1+x_{n,2}\mathbf{e}_2$.
$\ell^p(\mathscr{B}\mathbb{C})$	Componentwise p-summable bicomplex sequences.
$c_0(\mathscr{B}\mathbb{C}),c(\mathscr{B}\mathbb{C})$	Null and convergent bicomplex sequences.
$c_{00}(\mathscr{B}\mathbb{C})$	Finitely supported bicomplex sequences.
e_n	nth coordinate unit vector.
P_N	Projection onto the first N coordinates.
S,R	Right and left shift operators.

Symbol	Meaning
m_N(E)	Prefix-bounded sequence space generated by a multi-norm N.
X^×	Köthe dual of a sequence space X.

1. Introduction

Sequence spaces are test cases for functional analysis because their coordinate maps, bases, duals, matrix transformations, and compact operators can be written explicitly. Bicomplex sequence spaces preserve this accessibility while recording two complex channels. Recent work has examined completeness, geometry, topological duals, difference spaces, and Orlicz–lacunary constructions (Sager & Sağır, 2020; Değirmen & Sağır Duyar, 2022; Raj et al., 2023).

A bicomplex sequence x decomposes into two complex sequences. Consequently, summability and supremum conditions must be required componentwise. The hyperbolic norm retains both values and may lie in the null cone when only one component sequence is nonzero. Classical inclusion and duality results can then be recombined, but domain assumptions must be preserved. For example, $\ell^p \subset \ell q$ for $p \leq q$ is a counting-measure sequence result and reverses on finite-measure function spaces.

The second objective is to let a multi-norm generate a sequence space. Prefix norms $N_n(x_1,\ldots,x_n)$ are increasing, so their component supremum defines a natural bounded-family space. This construction connects multi-level geometry to shifts, tails, and operator-generated sequence spaces. The paper develops both classical and generated spaces, with proofs and counterexamples.

2. Definitions of Classical Bicomplex Sequence Spaces

Definition 2.1. A bicomplex sequence is x=(x_n) with x_n=x_{n,1}$\mathbf{e}_1$+x_{n,2}$\mathbf{e}_2$. Convergence x_n→x is componentwise complex convergence, equivalently |x_n−x|𝔻→0.

Definition 2.2. For 1≤p<∞,

$\ell^p(\mathscr{B}\mathbb{C})$={x:Σ_{n=1}∞|x_n|$\mathbb{D}^p$ converges componentwise}, (2.1)

‖x‖p,𝔻=(Σ_n|x_{n,1}|^p)^{1/p}$\mathbf{e}_1$+(Σ_n|x_{n,2}|^p)^{1/p}$\mathbf{e}_2$. (2.2)

Definition 2.3. The bounded sequence space is

$\ell\infty(\mathscr{B}\mathbb{C})$={x:sup_n|x_n|𝔻 exists finitely in $\mathbb{D}^+$}, (2.3)

with ‖x‖∞,𝔻=sup_n|x_n|𝔻, where the supremum is componentwise.

Definition 2.4. $c_0(\mathscr{B}\mathbb{C})$ consists of sequences converging to zero; $c(\mathscr{B}\mathbb{C})$ consists of all convergent sequences; $c_{00}(\mathscr{B}\mathbb{C})$ consists of finitely supported sequences. All use the supremum hyperbolic norm.

Definition 2.5. The coordinate projection π_n(x)=x_n, the truncation P_Nx=(x_1,…,x_N,0,…), the right shift Sx=(0,x_1,x_2,…), and the left shift Rx=(x_2,x_3,…) are bicomplex-linear operators.

Theorem 2.6. Each classical bicomplex sequence space decomposes isometrically:

$\ell^p(\mathscr{B}\mathbb{C})=\ell^p(\mathbb{C})\mathbf{e}_1 \oplus \ell^p(\mathbb{C})\mathbf{e}_2$, (2.4)

and likewise for ℓ∞, c_0, c, and c_{00}.

Proof. The defining sum, supremum, or limit exists in $\mathbb{D}^+$ exactly when it exists in each coefficient. Equations (2.2)–(2.3) are the corresponding component norms, so the decomposition is isometric. □

3. Completeness and Inclusion Relations

Theorem 3.1. The spaces $\ell^p(\mathscr{B}\mathbb{C})$ for $1\le p\le\infty$, $c_0(\mathscr{B}\mathbb{C})$, and $c(\mathscr{B}\mathbb{C})$ are bicomplex Banach modules.

Proof. By Theorem 2.6, a Cauchy sequence in any listed space is a pair of Cauchy sequences in the corresponding complex Banach spaces. Each component converges in that space, and recombination gives the bicomplex limit. □

Theorem 3.2. For counting-measure sequence spaces and $1\le p\le q\le\infty$,

$$\ell^p(\mathscr{B}\mathbb{C})\subset\ell q(\mathscr{B}\mathbb{C}),\quad \|x\|q,\mathbb{D}\preccurlyeq\mathbb{D}\|x\|p,\mathbb{D}. \tag{3.1}$$

Proof. For a complex sequence $a\in\ell^p$, sup_n|a_n|≤‖a‖p. If $q<\infty$, Σ|a_n|q=Σ|a_n|^{q−p}|a_n|p≤‖a‖∞^{q−p}‖a‖p^p≤‖a‖p^q. Taking qth roots gives ‖a‖q≤‖a‖p; q=∞ is the first inequality. Apply this to both idempotent components. □

Remark 3.3. Equation (3.1) depends on the counting-measure sequence domain. On a finite-measure function space, the usual inclusion direction is generally reversed. It should not be quoted without specifying the underlying measure structure.

Proposition 3.4. $c_0(\mathscr{B}\mathbb{C})$ is a closed submodule of $c(\mathscr{B}\mathbb{C})$, and $c(\mathscr{B}\mathbb{C})$ is a closed submodule of $\ell\infty(\mathscr{B}\mathbb{C})$. Moreover $c(\mathscr{B}\mathbb{C})=c_0(\mathscr{B}\mathbb{C})\oplus\mathscr{B}\mathbb{C}\cdot\mathbb{1}$, where $\mathbb{1}=(1,1,\ldots)$.

Proof. Each statement is the paired complex result. For the direct sum, a convergent x with limit L decomposes as $(x-L\mathbb{1})+L\mathbb{1}$, and the intersection is zero. □

Theorem 3.5. $c_{00}(\mathscr{B}\mathbb{C})$ is dense in $\ell^p(\mathscr{B}\mathbb{C})$ for $1\le p<\infty$ and in $c_0(\mathscr{B}\mathbb{C})$.

Proof. For x in ℓ^p, ‖x−P_Nx‖p,$\mathbb{D}$ has coefficients equal to the p-norms of the component tails, which tend to zero. For $x\in c_0$, the supremum of each tail tends to zero. Thus P_Nx$\in c_{00}$ and P_Nx→x. □

Counterexample 3.6. $c_{00}(\mathscr{B}\mathbb{C})$ is not dense in $\ell\infty(\mathscr{B}\mathbb{C})$.

Proof. For the constant sequence $\mathbb{1}$, every finitely supported y has |1−y_n|$\mathbb{D}$=1 for all sufficiently large n. Hence ‖$\mathbb{1}$−y‖∞,$\mathbb{D}\succcurlyeq\mathbb{D}$1, so no c_{00} sequence approximates $\mathbb{1}$. □

4. Coordinates, Shifts, and Bases

Proposition 4.1. For all classical spaces above, ‖π_n‖$\mathbb{D}$=1 and ‖P_N‖$\mathbb{D}$=1. On ℓ^p and c_0, ‖S‖$\mathbb{D}$=‖R‖$\mathbb{D}$=1, RS=I, and SR=I−P_1.

Proof. Coordinate magnitude is bounded by the relevant p- or supremum norm, with equality on e_n. Truncation does not increase any component norm. The right shift preserves component norms; the left shift is contractive and attains norm one. The algebraic identities follow by inspection. □

Definition 4.2. A Schauder basis {u_n} for a bicomplex Banach module X means every x has a unique norm-convergent expansion x=Σ_nα_nu_n, with bicomplex coefficients.

Theorem 4.3. The canonical unit vectors {e_n} form a bicomplex Schauder basis for $\ell^p(\mathscr{B}\mathbb{C})$, $1\le p<\infty$, and for $c_0(\mathscr{B}\mathbb{C})$.

Proof. The partial sums are P_Nx, which converge by Theorem 3.5. Uniqueness follows by applying coordinate projections: the nth coefficient must be x_n. Componentwise, this is the standard complex canonical basis. □

Example 4.4. For x_n=2^{-n}$\mathbf{e}_1$+3^{-n}$\mathbf{e}_2$, x belongs to every $\ell^p(\mathscr{B}\mathbb{C})$, $1\le p\le\infty$, and

$$\|x\|p,\mathbb{D}=[\Sigma\_n2\hat{}\{-np\}]\hat{}\{1/p\}\mathbf{e}_1+[\Sigma\_n3\hat{}\{-np\}]\hat{}\{1/p\}\mathbf{e}_2. \tag{4.1}$$

Proof. Both components are geometric series with ratios 2^{-p} and 3^{-p}. Their convergence proves membership; the displayed formula is the definition. □

Example 4.5. The unit vector e_n has $\|e_n\|_{p,\mathbb{D}}=1$ for every p. The right shift sends e_n to e_{n+1}, showing directly that S is an isometry.

Proof. Each component of e_n is the standard complex unit vector because its nonzero coordinate equals $1=\mathbf{e}_1+\mathbf{e}_2$. The norm and shift claims follow. □

5. Duals, Köthe Duals, and Weak Convergence

Definition 5.1. For a bicomplex sequence space X, its Köthe dual is

$$X^\times=\{y:\Sigma_n|x_ny_n|_{\mathbb{D}} \text{ converges componentwise} \tag{5.1}$$
$$\text{for every } x\in X\}.$$

Theorem 5.2. Let $1<p<\infty$ and $1/p+1/q=1$. Then

$$(\ell^p(\mathscr{B}\mathbb{C}))'\cong\ell^q(\mathscr{B}\mathbb{C}), \qquad (\ell^1(\mathscr{B}\mathbb{C}))'\cong\ell^\infty(\mathscr{B}\mathbb{C}), \tag{5.2}$$
$$c_0(\mathscr{B}\mathbb{C})'\cong\ell^1(\mathscr{B}\mathbb{C}),$$

isometrically in the hyperbolic norm, through $f_y(x)=\Sigma_n\ y_n^{\dagger 3}x_n$ under the chosen conjugate convention.

Proof. Decompose a bicomplex functional into two continuous complex functionals. Apply the classical dual identifications to obtain y_1 and y_2. Hölder's inequality proves boundedness, and evaluation on component unit vectors proves uniqueness and norm equality. Recombination gives Equation (5.2). □

Proposition 5.3. The Köthe duals are componentwise: $(\ell^p(\mathscr{B}\mathbb{C}))^\times=\ell^q(\mathscr{B}\mathbb{C})$ for $1\le p\le\infty$ with the usual conjugate conventions $q=\infty$ for $p=1$ and $q=1$ for $p=\infty$.

Proof. Absolute componentwise convergence of Σx_ny_n is precisely the pair of classical Köthe-dual conditions. Hölder gives sufficiency; standard extremal or truncation sequences give necessity. □

Theorem 5.4. For $X=c_0(\mathscr{B}\mathbb{C})$ or $\ell^p(\mathscr{B}\mathbb{C})$ with $1<p<\infty$, a sequence $x^{(m)}$ is weakly null if and only if it is norm bounded and every coordinate $x_n^{(m)}$ tends to zero in $\mathscr{B}\mathbb{C}$.

Proof. Weakly null sequences are bounded by uniform boundedness, and coordinate functionals give coordinate convergence. Conversely, decompose into components. For c_0, pair with ℓ^1 and split the dual sum into a finite head, controlled by coordinate convergence, and a small ℓ^1 tail, controlled by boundedness. The same proof for ℓ^p uses $y\in\ell^q$ with $q<\infty$. Recombine the two weak convergences. □

Counterexample 5.5. Bounded coordinatewise convergence does not imply weak convergence in $\ell^1(\mathscr{B}\mathbb{C})$.

Proof. The unit vectors e_m are bounded and each fixed coordinate tends to zero. Let $y=(1,1,\ldots)\in\ell^\infty(\mathscr{B}\mathbb{C})$, defining a continuous functional on ℓ^1. Then $f_y(e_m)=1$ for every m, so e_m is not weakly null. This is the bicomplex form of the ℓ^1 obstruction. □

6. Multi-Norm-Generated Sequence Spaces

Definition 6.1. Let E be a bicomplex multi-normed module with $N=\{N_n\}$. The prefix-bounded space is

$$m_N(E)=\{x=(x_k)\subset E: \tag{6.1}$$
$$\|x\|_{m_N,\mathbb{D}}:=\sup_{n\ge1}N_n(x_1,\ldots,x_n)<\infty$$
$$\text{componentwise}\}.$$

The prefix values are increasing because appending a coordinate dominates appending zero. Hence the component supremum is well defined in $[0,\infty]$.

Theorem 6.2. $m_N(E)$ is a bicomplex normed module. If E is complete, then $m_N(E)$ is complete.

Proof. Homogeneity and the triangle inequality pass through component suprema. If $\|x\|_{m_N,\mathbb{D}}=0$, then $N_n(x_1,\ldots,x_n)=0$ for every n, so every coordinate is zero. For completeness, let $x^{(r)}$ be Cauchy in m_N. Since $\|x_k^{(r)}-x_k^{(s)}\|_{\mathbb{D}}\preccurlyeq_{\mathbb{D}}\|x^{(r)}-x^{(s)}\|_{m_N}$, each coordinate converges in E to x_k. Fix r beyond the Cauchy index and a level n. Passing $s\to\infty$ in $N_n(x_1^{(r)}-x_1^{(s)},\ldots,x_n^{(r)}-x_n^{(s)})$ gives a bound by ε. Because this holds for every n, the supremum is at most ε. Thus $x^{(r)}\to x$ in m_N and x belongs to the space. □

Definition 6.3. The tail-vanishing space is

$$c_{0,N}(E)=\{x\in m_N(E): \lim_{r\to\infty} \sup_{m\geq 1} N_m(x_{r+1},\dots,x_{r+m})=0\}. \quad (6.2)$$

Proposition 6.4. $c_{0,N}(E)$ is a closed submodule of $m_N(E)$. For the minimum multi-norm, $m_N(E)=\ell^\infty(E)$ and $c_{0,N}(E)=c_0(E)$.

Proof. Linearity follows from the tail triangle inequality. If $x^{(s)}\to x$ in m_N and $x^{(s)}$ has small tails, then the tail of x is bounded by the m_N-distance to $x^{(s)}$ plus the tail of $x^{(s)}$, proving closedness. For $N_n^{\min}=\max_{k\leq n}\|x_k\|$, the prefix supremum is $\sup_k\|x_k\|$ and the tail expression is $\sup_{k>r}\|x_k\|$, which tends to zero exactly for a null sequence. □

Example 6.5. If $E=\mathscr{B}\mathbb{C}$ with the minimum multi-norm, the generated space $m_N(E)$ is exactly $\ell^\infty(\mathscr{B}\mathbb{C})$. Thus the constant sequence $\mathbb{1}$ is prefix bounded but not tail vanishing.

Proof. Proposition 6.4 gives the identification. Every prefix maximum of $\mathbb{1}$ equals 1, while every tail supremum also equals 1 and does not tend to zero. □

Example 6.6. For a Hilbert multi-norm N^H on E, $m_{N^H}(E)$ can be strictly smaller than $\ell^\infty(E)$: bounded coordinates may accumulate through mutually orthogonal projections.

Proof. Take an infinite-dimensional complex component with orthonormal vectors u_k and set $x_k=u_k$ in that component. Choosing P_k as projection onto $\mathrm{span}\{u_k\}$ gives $N_n^H(x_1,\dots,x_n)\geq\|\Sigma_{k=1}^n u_k\|=\sqrt{n}$, so the sequence is not prefix bounded although every coordinate has norm one. The other bicomplex component can be chosen identically or zero. □

7. Matrix Transformations and Operator-Generated Spaces

Proposition 7.1. Let $A=(a_{nk})$ be a bicomplex matrix. If

$$C_\infty=\sup_n \Sigma_k|a_{nk}|_{\mathbb{D}}<\infty, \quad (7.1)$$

then Ax with $(Ax)_n=\Sigma_k a_{nk}x_k$ defines a bounded operator on $\ell^\infty(\mathscr{B}\mathbb{C})$ with $\|A\|_{\mathbb{D}}\preccurlyeq_{\mathbb{D}}C_\infty$. If $C_1=\sup_k\Sigma_n|a_{nk}|_{\mathbb{D}}<\infty$, A is bounded on ℓ^1 with norm at most C_1.

Proof. For ℓ^∞, $|(Ax)_n|_{\mathbb{D}}\preccurlyeq_{\mathbb{D}}\Sigma_k|a_{nk}|_{\mathbb{D}}\|x\|_{\infty,\mathbb{D}}$; take the component supremum over n. For ℓ^1, sum over n, interchange nonnegative component sums, and use the column bound. □

Example 7.2. The diagonal matrix $a_{nk}=d_n\delta_{nk}$ defines $D_dx=(d_nx_n)$. It is bounded on every ℓ^p when d is bounded, with $\|D_d\|_{\mathbb{D}}=\sup_n|d_n|_{\mathbb{D}}$, and compact for $1\leq p<\infty$ when $d_n\to 0$ componentwise.

Proof. The norm formula is immediate from multiplication and unit vectors. If $d_n\to 0$, finite-coordinate truncations are finite rank and converge to D_d in operator norm. Conversely, compactness applied to e_n forces $\|D_de_n\|=|d_n|_{\mathbb{D}}\to 0$, since disjoint unit vectors cannot have a convergent image subsequence unless their magnitudes vanish. □

Theorem 7.3. Coordinate-tail compactness criterion. Let X be $c_0(\mathscr{B}\mathbb{C})$ or $\ell^p(\mathscr{B}\mathbb{C})$, $1\leq p<\infty$. A bounded $T:X\to X$ is compact if and only if

$$\sup_{\|x\|_{\mathbb{D}}\preccurlyeq_{\mathbb{D}}1} \|(I-P_N)Tx\|_{\mathbb{D}}\to 0 \quad (7.2)$$

componentwise.

Proof. If the tail condition holds, P_NT is finite rank and converges to T in operator norm, so T is compact. Conversely, T maps the unit ball to a relatively compact set. The truncations P_N converge pointwise to I and are uniformly bounded; on a compact set this convergence is uniform by a finite-net argument in each component. Therefore the displayed supremum tends to zero. □

Definition 7.4. Given bicomplex-linear maps $T_n:E\to F$, define the operator-generated space

$$\ell^p(T_n;E,F)=\{x=(x_n): (T_nx_n)\in\ell^p(F)\}, \quad (7.3)$$

$$\|x\|_{T,p}=\|(T_nx_n)\|_p.$$

It is a normed space when the family separates coordinates; uniform lower bounds make it equivalent to $\ell^p(E)$.

Proposition 7.5. If $a,b\in\mathbb{D}^+$ have positive coefficients and $a\|x\|E\preccurlyeq\mathbb{D}\|T\_nx\|F\preccurlyeq\mathbb{D}b\|x\|E$ for every n and x, then $\ell^p(T\_n;E,F)=\ell^p(E)$ with

$$a\|x\|p,E\preccurlyeq\mathbb{D}\|x\|T,p\preccurlyeq\mathbb{D}b\|x\|p,E. \qquad (7.4)$$

Proof. Raise the component inequalities to the pth power, sum, and take component roots; for $p=\infty$ take suprema. The positive lower coefficients guarantee definiteness and equivalence. □

7.6 Cesàro Operators, Shift Spectra, and Matrix Domains

Example 7.6. The bicomplex Cesàro operator C on sequences is

$$(Cx)\_n=(1/n)\Sigma\_\{k=1\}^n\ x\_k. \qquad (7.5)$$

It is bounded on $\ell\infty(\mathscr{B}\mathbb{C})$, $c(\mathscr{B}\mathbb{C})$, and $c_0(\mathscr{B}\mathbb{C})$ with $\|C\|\mathbb{D}=1$, and it preserves limits.

Proof. Every row has nonnegative absolute row sum one, so Proposition 7.1 gives $\|C\|\leq 1$; equality follows from the constant sequence. Componentwise Cesàro convergence preserves an existing limit and sends a null sequence to a null sequence. □

Proposition 7.7. The Cesàro operator is not compact on $c_0(\mathscr{B}\mathbb{C})$.

Proof. Let x^(m) be the sequence equal to one in the first m positions and zero thereafter. These vectors lie in the unit ball of c_0. For fixed n, $(Cx^{(m)})\_n\to 1$ as $m\to\infty$. If a subsequence converged in c_0 norm, its pointwise limit would be the constant sequence one, which is not in c_0. Hence the image of the unit ball is not relatively compact. □

Theorem 7.8. Reduced spectrum of the unilateral shift. On $\ell^p(\mathscr{B}\mathbb{C})$, $1\leq p<\infty$, or $c_0(\mathscr{B}\mathbb{C})$, the right shift S has

$$\sigma red(S)=\{z:|z|\leq 1\}\mathbf{e}_1+\{z:|z|\leq 1\}\mathbf{e}_2. \qquad (7.6)$$

The full bicomplex spectrum is the union of the two cylinders generated by the closed unit disk.

Proof. Each component is the classical unilateral shift, whose spectrum is the closed unit disk. Apply the reduced paired definition and the full spectral-cylinder formula. This distinction matters: the full spectrum contains values with one arbitrary complex coordinate whenever the other lies in the disk. □

Definition 7.9. For a matrix A acting on sequences and a bicomplex sequence space X, the matrix domain is $X\_A=\{x:Ax\in X\}$, equipped with the graph norm $\|x\|A,\mathbb{D}=\|x\|base,\mathbb{D}+\|Ax\|X,\mathbb{D}$ on a selected base domain.

Proposition 7.10. If the component matrix operators A_j are closed from the component base spaces into X_j, then X_A is complete under the graph norm whenever the base and target spaces are Banach.

Proof. A graph-norm Cauchy sequence is Cauchy both in the base space and after applying A. Component completeness gives x and y; closedness of A_j yields x in the domain and Ax=y. Recombine the two component limits. □

Proposition 7.11. If a multi-norm N is translation invariant on tails in the sense that the same level norms are used for every block, then the left shift R is contractive on m_N(E) and on c_{0,N}(E).

Proof. Every prefix of Rx is a block $(x_2,\ldots,x\_\{n+1\})$. By scalar domination and the standard multi-norm lower/upper structure, the assumed block invariance bounds its N_n value by the supremum defining $\|x\|m\_N$. Tail vanishing is preserved because shifting merely advances the tail index. The hypothesis is essential: a position-weighted family need not be shift invariant. □

Example 7.12. For the minimum multi-norm, the hypothesis of Proposition 7.11 holds and R is a contraction on $\ell\infty(E)$ and $c_0(E)$. For a weighted prefix gauge that multiplies the kth coordinate by an increasing position weight before taking a maximum, the left or right shift can change the norm and may be unbounded.

Proof. The minimum block norm depends only on coordinate norms, not on positions. In a weighted gauge, moving a vector into a position with a larger weight amplifies it. Such a gauge may fall outside the standard multi-norm axioms unless the weights also respect repetition and zero stability, illustrating why shift questions must be tied to a precise construction. □

Cesàro and shift operators show how classical sequence transforms acquire two spectral and norm channels. The matrix-domain construction is useful for difference operators and Orlicz–lacunary spaces: completeness is a closed-graph question in each component. Multi-norm-generated spaces add block geometry, and shift invariance is no longer automatic unless the multi-norm treats positions uniformly.

These results also sharpen computational interpretation. Finite truncations approximate compact diagonal operators but not the Cesàro operator on c_0. The right shift is norm preserving yet has a reduced disk spectrum and a full cylindrical spectrum. Algorithms that use bicomplex sequence operators should therefore report whether they analyze component, reduced, or full spectral data.

8. Applications and Interpretive Analysis

Pairs of discrete signals are represented by bicomplex sequence spaces. Delays are represented by shifts, coordinate filters by diagonal operators, and linear transformations by matrices. Separate advances are reported by hyperbolic norms. The signal is not zero; rather, a null-cone norm indicates that only one channel is active.

All finite prefixes are subject to control in multi-norm-generated spaces. This is ordinary boundedness under the minimum multi-norm; however, it can limit the number of coordinates that occupy mutually separated directions under Hilbert or other stronger multi-norms. Vector-valued time series, frame coefficients, or batches of learned representations may benefit from such spaces; however, the intended geometry must be matched by the specific multi-norm.

Summability methods and sequence transformations are supported by Köthe duals and matrix criteria. Preprocessing or observation maps T_n are encoded in operator-generated spaces. Finite-coordinate approximation is justified by compactness criteria, and coordinate disappearance is distinguished from true functional convergence by weak convergence results. Despite the fact that these tools are directly computational, any purported application advantage necessitates a concrete model and data analysis.

9. Open Problems and Limitations

The prefix space m_N is a natural choice for multi-norm-generated sequence construction; however, it is not the sole option. Different spaces may be generated by weighted block norms, Köthe echelon limits, lacunary approaches, and Orlicz functions. The selected multi-norm can have a significant impact on the duals and compactness criteria.

Open problems encompass a general duality theory for m_N and $c_{0,N}$, matrix-domain characterisations, weak compactness and reflexivity, Schauder decompositions for unequal component structures, interpolation, spectra of shifts and band operators under full bicomplex invertibility, and multi-norm versions of summing and nuclear maps. Candidate contexts are provided by recent Orlicz–lacunary and geometric bicomplex sequence research; however, zero-divisor and order conventions must be explicitly stated.

10. In summary

Classical bicomplex sequence spaces are isometric pairs of complex sequence spaces that produce precise results on completeness, inclusions, density, coordinates, shifts, bases, duals, Köthe duals, weak convergence, matrices, and compactness. The inclusion ℓ^p-to-ℓ^q for $p \le q$ is valid on the counting-measure sequence domain and is component-wise valid. The prefix space $m_N(E)$ generated by the multi-norm is Banach when E is Banach, and its tail subspace generalises c_0. The minimum multi-norm recovers ℓ^∞ and c_0, while the Hilbert multi-norm can impose stronger geometric restrictions. Counterexamples distinguish coordinatewise from weak convergence and finite-support density from the $\ell\infty$ topology. Collectively, these findings establish a rigorous foundation for matrix transformations, operator-generated spaces, and future bicomplex sequence-space analysis.

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