Hyperbolic-Valued Multi-Norms: Theory, Examples, and Emerging Applications

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Abstract:
A two-component method for measuring finite families in modules over the split-complex algebra is provided by hyperbolic-valued multi-norms. Their usefulness is contingent upon the availability of concrete constructions whose cross-level axioms can be verified, rather than on formal notation. This paper establishes a theory that emphasises the use of examples. It constructs minimum, Hilbert-projection, function-space, sequence-space, pullback, and operator-family multi-norms by idempotent recombination after reviewing the classical multi-norm axioms and the positive hyperbolic cone. Permutation invariance, coordinatewise scalar domination, stability under adjoining zero, repetition consistency, and compatibility with the underlying norm are all verified for each proposed construction. In a common hierarchy, comparison estimates arrange maximum, sum, and Euclidean tuple gauges. The latter two are demonstrated to be genuine norms on each finite power, but they are not multi-norms per the standard repetition axiom. The proof of fixed-level equivalence in finite dimensions is presented, and the level-dependent nature of equivalence constants is underscored. Null-cone values and asymmetric component geometry are illustrated in the worked calculations. Interpretive applications are created to address the stability of coupled systems, split-channel optimisation, uncertain models, operator equations, and hypercomplex machine-learning representations. The discussion does not assert empirical superiority; rather, it identifies mathematically testable responsibilities for multi-level control. The following are open problems: uniform equivalence, computational approximations of Hilbert multi-norms, coupled nondecomposable constructions, and maximum multi-norms.
Keywords: hyperbolic multi-norm; minimum multi-norm; Hilbert multi-norm; function space; sequence space; split-complex model; stability

Mathematical Notation and Symbols

Symbol	Meaning
$\mathbb{D}, \mathbb{D}^+$	Hyperbolic algebra and its positive idempotent cone.
$\mathbf{e}_1, \mathbf{e}_2$	Orthogonal idempotents with $\mathbf{e}_1\mathbf{e}_2=0$ and $\mathbf{e}_1+\mathbf{e}_2=1$.
N_n	A level-n $\mathbb{D}^+$-valued multi-norm.
N_n^min	Minimum, or coordinate-maximum, multi-norm.
N_n^H	Hilbert projection multi-norm.
S_n, Q_n	Sum-type and Euclidean-type tuple gauges.
Γ	A norming family of hyperbolic-linear operators.

Symbol	Meaning
$C(K,\mathbb{D})$, $\ell^p(\mathbb{D})$	Hyperbolic-valued continuous-function and sequence modules.

1. Introduction

Abstract axioms are useful only when they discriminate among constructions. For a family of tuple norms, permutation invariance and the triangle inequality are easy to obtain; the more restrictive requirements are stability under adjoining zero and consistency under repetition. The latter axiom ensures that repeating an existing vector does not manufacture new size merely because the tuple is longer. Classical multi-norm theory was designed around this principle (Dales & Polyakov, 2012).

Over the hyperbolic algebra, every value has two idempotent coefficients. The order is therefore componentwise, and a valid construction can combine two classical multi-norms without comparing their coefficients. This gives a broad supply of examples, including component spaces with different geometries. It also creates null-cone norm values: a nonzero tuple supported in one idempotent component has a nonzero but noninvertible multi-norm. Such values are expected, not defective.

The paper emphasizes six constructions and verifies their axioms rather than appealing to analogy. It then compares them with sum and Euclidean tuple norms, proves finite-dimensional fixed-level equivalence, and evaluates emerging applications. The term "emerging" is used descriptively: applications are proposed mathematical interpretations unless linked to published hypercomplex models.

2. Foundations and Axioms

Definition 2.1. The hyperbolic algebra is $\mathbb{D}=\mathbb{R}\mathbf{e}_1\oplus\mathbb{R}\mathbf{e}_2$ with $\mathbf{e}_j^2=\mathbf{e}_j$, $\mathbf{e}_1\mathbf{e}_2=0$. Its positive cone is $\mathbb{D}^+=\{u_1\mathbf{e}_1+u_2\mathbf{e}_2:u_j\geq 0\}$, ordered by $u\preccurlyeq_{\mathbb{D}} v$ iff $u_j\leq v_j$ for $j=1,2$.

Definition 2.2. Let $E=\mathbf{e}_1E_1\oplus\mathbf{e}_2E_2$. A decomposable hyperbolic multi-norm is

$$N_n(x_1,\dots,x_n)=N_{n,1}(x_{11},\dots,x_{n1})\mathbf{e}_1+N_{n,2}(x_{12},\dots,x_{n2})\mathbf{e}_2, \quad (2.1)$$

where each $\{N_{n,j}\}$ is a classical multi-norm. Equation (2.1) automatically converts every real inequality into a $\mathbb{D}$-order inequality. The four level axioms are denoted (M1) permutation, (M2) scalar domination, (M3) zero stability, and (M4) repetition consistency.

Proposition 2.3. A map defined by Equation (2.1) satisfies all multi-norm axioms, and every axiom can be verified by projecting onto the two idempotents.

Proof. For (M1), permuting the hyperbolic tuple permutes both component tuples and leaves each $N_{n,j}$ unchanged. For (M2), write $\alpha_r=\alpha_{r1}\mathbf{e}_1+\alpha_{r2}\mathbf{e}_2$ and apply the real scalar-domination axiom separately. For (M3), an appended hyperbolic zero appends zero to both component tuples. For (M4), a repeated hyperbolic vector repeats both component vectors. Definiteness and the triangle inequality are also paired component statements. □

Definition 2.4. For any tuple, define the auxiliary gauges

$$M_n=\max_{1\leq r\leq n}\|x_r\|_{\mathbb{D}},\qquad S_n=\Sigma_r\|x_r\|_{\mathbb{D}}, \quad (2.2)$$
$$Q_n=(\Sigma_r\|x_r\|_{\mathbb{D}}^2)^{1/2},$$

where powers, square roots, sums, and maxima are componentwise. M_n will be a multi-norm; S_n and Q_n serve as comparison gauges and counterexamples.

Proposition 2.5. For every n and tuple,

$$M_n\preccurlyeq_{\mathbb{D}}Q_n\preccurlyeq_{\mathbb{D}}S_n\preccurlyeq_{\mathbb{D}}nM_n, \quad\text{and}\quad Q_n\preccurlyeq_{\mathbb{D}}\sqrt{n}\,M_n. \quad (2.3)$$

Proof. Each coefficient is the corresponding inequality among nonnegative real numbers: maximum≤Euclidean norm≤sum≤n·maximum and Euclidean norm≤√n·maximum. Recombine the two coefficients. □

3. Canonical and Maximum-Type Constructions

Definition 3.1. The minimum hyperbolic multi-norm associated with $\|\cdot\|_{\mathbb{D}}$ is $N_n\text{^}min(x_1,\ldots,x_n)=\max_r\|x_r\|_{\mathbb{D}}$.

Theorem 3.2. The family $\{N_n\text{^}min\}$ is a hyperbolic multi-norm and is pointwise no larger than any other hyperbolic multi-norm inducing the same N_1.

Proof. Permutation leaves a maximum unchanged. For scalars α_r, $\max_r\|\alpha_r x_r\|_{\mathbb{D}}\preccurlyeq_{\mathbb{D}}(\max_r|\alpha_r|_{\mathbb{D}})(\max_r\|x_r\|_{\mathbb{D}})$. Appending zero or repeating a coordinate does not change the maximum, proving (M3) and (M4). For minimality, any multi-norm N_n satisfies $\|x_r\|_{\mathbb{D}}\preccurlyeq_{\mathbb{D}}N_n(x_1,\ldots,x_n)$ by zeroing the other coordinates and using permutation and zero stability. Taking the component maximum over r gives $N_n\text{^}min\preccurlyeq_{\mathbb{D}}N_n$. □

Example 3.3. Finite-dimensional maximum calculation. Let $E=\mathbb{D}^2$ with $\|(z,w)\|_{\mathbb{D}}=\max(|z_1|,|w_1|)\mathbf{e}_1+\max(|z_2|,|w_2|)\mathbf{e}_2$. For $x=(1,-2)\mathbf{e}_1+(3,0)\mathbf{e}_2$ and $y=(-4,1)\mathbf{e}_1+(1,5)\mathbf{e}_2$, $N_2\text{^}min(x,y)=4\mathbf{e}_1+5\mathbf{e}_2$.

Proof. The first component norms are 2 and 4; the second are 3 and 5. Their component maxima give $4\mathbf{e}_1+5\mathbf{e}_2$. Permutation, scaling, zero, and repetition properties are exactly those verified in Theorem 3.2. □

Definition 3.4. When E_j are Hilbert spaces, the component Hilbert multi-norm is

$$N_n,j\text{^}H(x_1,\ldots,x_n)=\sup\{\|\Sigma_r P_r x_r\|:\ P_r \text{ are mutually orthogonal projections},\ \Sigma_r P_r=I\}. \tag{3.1}$$

The hyperbolic Hilbert multi-norm is $N_n\text{^}H=N_n,1\text{^}H\mathbf{e}_1+N_n,2\text{^}H\mathbf{e}_2$.

Theorem 3.5. The hyperbolic Hilbert construction is a multi-norm.

Proof. It suffices to verify the established component construction. Permuting vectors and projections together leaves the supremum unchanged. For coordinate scalars, decompose $\Sigma P_r\alpha_r x_r$ and use the operator defined by $\Sigma\alpha_r P_r$, whose norm is $\max_r|\alpha_r|$, to obtain scalar domination. Appending zero allows an additional projection with zero contribution; conversely merge that projection into a neighboring projection, proving equality. When the last vector is repeated, merge the last two orthogonal projections because $(P_{n-1}+P_n)x_{n-1}$ gives the same sum; conversely split a projection, proving repetition consistency. Recombine through Equation (2.1). □

Example 3.6. In one-dimensional component Hilbert spaces, every orthogonal resolution has at most one nonzero projection. Therefore $N_n,j\text{^}H(x_1,\ldots,x_n)=\max_r|x_r|$, and the hyperbolic Hilbert and minimum multi-norms coincide on $E=\mathbb{D}$.

Proof. A projection on a one-dimensional Hilbert space is either zero or identity. A family of mutually orthogonal projections summing to identity has exactly one identity member, so the supremum selects the largest coordinate magnitude. The four axioms reduce to those of the maximum. □

4. Function-, Sequence-, Pullback-, and Operator-Generated Examples

Definition 4.1. For compact K, let $E=C(K,\mathbb{D})$. Define the uniform family multi-norm

$$N_n\text{^}K(f_1,\ldots,f_n)=\sup\_\{t\in K\}\ \max_r|f_{r1}(t)|\ \mathbf{e}_1+\sup\_\{t\in K\}\ \max_r|f_{r2}(t)|\ \mathbf{e}_2. \tag{4.1}$$

Proposition 4.2. The maps $N_n\text{^}K$ form a complete hyperbolic multi-normed structure.

Proof. For each component, $N_n,j\text{^}K=\max_r\|f_{rj}\|_\infty$, so permutation, scalar domination, zero stability, and repetition are those of the minimum multi-norm. The level-one norm is the component supremum norm. Since $C(K,\mathbb{R})$ is Banach, both component finite powers are complete, and hence so is the hyperbolic product. □

Example 4.3. Let $K=[0,1]$, $f(t)=t\mathbf{e}_1+t^2\mathbf{e}_2$ and $g(t)=(1-t)\mathbf{e}_1+2t\mathbf{e}_2$. Then $N_2\text{^}K(f,g)=1\mathbf{e}_1+2\mathbf{e}_2$.

Proof. For the first component, sup_t $\max(t,1-t)=1$. For the second, sup_t $\max(t^2,2t)=2$. The value is therefore $\mathbf{e}_1+2\mathbf{e}_2$. Appending zero or repeating either function leaves the pointwise maximum, and hence its supremum, unchanged. □

Definition 4.4. On $\ell^p(\mathbb{D})=\mathbf{e}_1\ell^p(\mathbb{R})\oplus\mathbf{e}_2\ell^p(\mathbb{R})$, $1\le p\le\infty$, define $N_n^{\{p,\min\}}(x^1,\dots,x^n)=\max_r\|x^r\|_{p,\mathbb{D}}$.

Proposition 4.5. The sequence-space construction is a complete hyperbolic multi-norm, and every coordinate projection π_k is multi-contractive.

Proof. The first assertion is Theorem 3.2 combined with completeness of the component ℓ^p spaces. For π_k, $|x\_k^r|_{\mathbb{D}}\preccurlyeq_{\mathbb{D}}\|x^r\|_{p,\mathbb{D}}$ for each r, so $\max_r|\pi\_kx^r|_{\mathbb{D}}\preccurlyeq_{\mathbb{D}}\max_r\|x^r\|_{p,\mathbb{D}}$ at every level. □

Example 4.6. For $p=2$, $x^1=(\mathbf{e}_1,2\mathbf{e}_2,0,\dots)$ and $x^2=(2\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_1+\mathbf{e}_2,0,\dots)$, one obtains $N_2^{\{2,\min\}}(x^1,x^2)=\sqrt{5}\mathbf{e}_1+\sqrt{5}\mathbf{e}_2$.

Proof. The component norms of x^1 are 1 and 2. Those of x^2 are $\sqrt{(4+1)}=\sqrt{5}$ and $\sqrt{(1+1)}=\sqrt{2}$. Taking component maxima gives $\sqrt{5}$ in both channels. The zero and repetition axioms follow because the outer aggregation is a maximum over the tuple members. □

Definition 4.7. Pullback construction. If $J:E\to F$ is an injective hyperbolic-linear map and F has a hyperbolic multi-norm M with $M_1(Jx)=\|x\|_{\mathbb{D}}$, define $N_n^J(x_1,\dots,x_n)=M_n(Jx_1,\dots,Jx_n)$.

Proposition 4.8. The pullback construction is a hyperbolic multi-norm.

Proof. Every level axiom follows by applying the corresponding axiom for M after J. The triangle inequality and homogeneity follow from linearity. If $N_n^J(x)=0$, definiteness of M gives $Jx_r=0$ for all r; injectivity gives $x_r=0$. The level-one compatibility is assumed. □

Example 4.9. Let $J:\mathbb{D}^2\to C([0,1],\mathbb{D})$ be $J(a,b)(t)=a+tb$, and let $M=N^K$. Then J is injective and the induced N_n^J measures the largest uniform affine response in each idempotent channel.

Proof. If $a+tb=0$ for all t, evaluation at $t=0$ gives $a=0$ and then $b=0$, so J is injective. All four axioms are inherited from N^K by Proposition 4.8. For a single pair (a,b), the underlying norm is $\sup_{t\in[0,1]}|a_j+tb_j|$ in each component. □

Definition 4.10. Operator-family construction. Let Γ be a nonempty family of hyperbolic-linear maps $T:E\to F$ with uniformly bounded component operator norms, and suppose Γ is norming: $\|x\|_{\mathbb{D}}=\sup_{T\in\Gamma}\|Tx\|_{\mathbb{D}}$. Define

$$N_n^\Gamma(x_1,\dots,x_n)=\sup_{T\in\Gamma} M_n(Tx_1,\dots,Tx_n), \qquad (4.2)$$

where the supremum is componentwise and finite.

Theorem 4.11. If M is a hyperbolic multi-norm on F and the supremum in Equation (4.2) is finite, then N^Γ is a hyperbolic multi-norm.

Proof. Permutation, scalar domination, zero stability, and repetition hold for each T and pass to componentwise suprema. For the triangle inequality, $M_n(T(x+y))\preccurlyeq_{\mathbb{D}}M_n(Tx)+M_n(Ty)$; taking sup gives at most the sum of the two suprema. Definiteness follows at level one from the norming condition, and for a tuple from the lower estimate $\max_r\|x_r\|_{\mathbb{D}}\preccurlyeq_{\mathbb{D}}N_n^\Gamma$. Compatibility at $n=1$ is exactly the norming identity. □

Example 4.12. Take $\Gamma=\{I,P_1,P_2\}$ on E, where $P_jx=\mathbf{e}_jx$, and let $M=N^{\min}$. Then $N_n^\Gamma=N_n^{\min}$.

Proof. The identity gives $N_n^\Gamma\succcurlyeq_{\mathbb{D}}N_n^{\min}$. Every projection is contractive componentwise, so its contribution is no larger than $N_n^{\min}$. The supremum is therefore equality. Each axiom is inherited through Theorem 4.11. □

5. Sum-Type and Euclidean-Type Failures

Counterexample 5.1. The sum-type gauge S_n in Equation (2.2) is not a multi-norm whenever E contains a nonzero vector.

Proof. Let $x\neq0$. Although S_n is permutation invariant, homogeneous, positive, and stable under appending zero, repetition gives $S_2(x,x)=2\|x\|_{\mathbb{D}}$ whereas $S_1(x)=\|x\|_{\mathbb{D}}$. At least one component is positive, so equality fails. Normalizing by n would repair this calculation but destroy zero stability: S_n/n changes when a zero is appended. □

Counterexample 5.2. The Euclidean-type gauge Q_n is not a multi-norm under the standard repetition axiom.

Proof. For $x\neq 0$, $Q_2(x,x)=\sqrt{2}\|x\|_{\mathbb{D}}\neq Q_1(x)$. It remains a useful norm on E^n and satisfies the inequalities in Proposition 2.5, but it belongs to a different class of power-type tuple norms. Calling it a multi-norm without changing the axioms would be mathematically incorrect. □

6. Comparison and Finite-Dimensional Equivalence

Theorem 6.1. Let N and M be decomposable hyperbolic multi-norms on a finite-dimensional E. For each fixed n there are invertible positive hyperbolic constants c_n, C_n such that $c_n N_n \preccurlyeq_{\mathbb{D}} M_n \preccurlyeq_{\mathbb{D}} C_n N_n$.

Proof. The component spaces E_j^n are finite dimensional. The unit sphere of N_n,j is compact, and the continuous function M_n,j has a positive minimum and finite maximum on it. These constants yield real norm equivalence in each component. Recombining gives $c_n=c_n,1\mathbf{e}_1+c_n,2\mathbf{e}_2$ and $C_n=C_n,1\mathbf{e}_1+C_n,2\mathbf{e}_2$, both with positive coefficients. □

Table 1. Comparison of representative hyperbolic tuple constructions.

Construction	Valid multi-norm?	Repetition behavior	Primary use
Minimum / coordinate maximum	Yes	Duplicate leaves maximum unchanged	Baseline and operator-stable structure
Hilbert projection	Yes	Merge the two projections on a repeated vector	Orthogonal geometry
Function-space supremum maximum	Yes	Pointwise maximum unchanged	Uniform control of function families
Sequence-space minimum	Yes	Outer maximum unchanged	Coordinate and shift estimates
Raw sum S_n	No	$S_2(x,x)=2S_1(x)$	Comparison gauge only
Euclidean tuple Q_n	No	$Q_2(x,x)=\sqrt{2}Q_1(x)$	Power-type tuple norm

Remark 6.2. Theorem 6.1 is fixed-level. A uniform constant valid for every n is a much stronger statement and may fail even in finite-dimensional base spaces because the multi-norm sequence can encode level-dependent growth.

6.3 Orthogonal Geometry and Stability Examples

Theorem 6.3. Orthogonal-tuple value of the Hilbert multi-norm. Suppose $x_1,\ldots,x_n$ are pairwise orthogonal in both idempotent Hilbert components. Then

$$N_n^H(x_1,\ldots,x_n)=(\Sigma_r\|x_r\|_{\mathbb{D}}^2)^{1/2}. \qquad (6.1)$$

Proof. For any orthogonal resolution $\{P_r\}$, the vectors $P_r x_r$ lie in mutually orthogonal ranges, so $\|\Sigma P_r x_r\|^2=\Sigma\|P_r x_r\|^2\leq\Sigma\|x_r\|^2$ in each component. For the reverse inequality, choose P_r as projection onto span$\{x_r\}$ for nonzero x_r and place the remaining orthogonal complement into any projection. Then $P_r x_r=x_r$, attaining the bound. Recombine componentwise. □

Example 6.4. Let E=$\mathscr{B}\mathbb{C}$-like hyperbolic Hilbert module with orthonormal u,v and $x_1=2u\mathbf{e}_1+v\mathbf{e}_2$, $x_2=u\mathbf{e}_1+3v\mathbf{e}_2$ arranged so that the corresponding component vectors are orthogonal. Then $N_2^H(x_1,x_2)=\sqrt{5}\mathbf{e}_1+\sqrt{10}\mathbf{e}_2$, whereas N_2^min$=2\mathbf{e}_1+3\mathbf{e}_2$.

Proof. The first component squared norms are 4 and 1; the second are 1 and 9. Theorem 6.3 gives the square-root sums. The minimum multi-norm takes the larger individual norm in each component. This explicitly shows that Hilbert family geometry can exceed worst-coordinate geometry. □

Proposition 6.5. For every hyperbolic Hilbert tuple,

N_n^min$\preccurlyeq\mathbb{D}N_n^H\preccurlyeq\mathbb{D}(\Sigma_r\|x_r\|\mathbb{D}^2)$^{1/2}$\preccurlyeq\mathbb{D}\Sigma_r\|x_r\|\mathbb{D}$. (6.2)

Proof. The first inequality is the universal minimum bound. For any orthogonal projections, Pythagoras gives $\|\Sigma P_r x_r\|^2=\Sigma\|P_r x_r\|^2\leq\Sigma\|x_r\|^2$; take the supremum. The final inequality is the component ℓ^2–ℓ^1 comparison. □

Theorem 6.6. Paired contraction stability. Let F:E→E satisfy $N_n(Fx_1-Fy_1,\ldots,Fx_n-Fy_n)\preccurlyeq\mathbb{D}L$ $N_n(x_1-y_1,\ldots,x_n-y_n)$ for all n, where $L=L_1\mathbf{e}_1+L_2\mathbf{e}_2$ and $0\leq L_j<1$. If E is complete, F has a unique fixed point and

$\|F^m x-x^*\|\mathbb{D}\preccurlyeq\mathbb{D}L^m(1-L)^{-1}\|Fx-x\|\mathbb{D}$. (6.3)

Proof. At n=1 the component maps are contractions with constants L_j. Banach's fixed-point theorem yields unique component fixed points x_j, *hence* $x=x_1\boldsymbol{e}_1+x_2\mathbf{e}_2$. The standard geometric error estimates hold in each component; because both $1-L_j$ are positive, $(1-L)^{-1}$ exists and recombination gives Equation (6.3). The family inequality additionally controls simultaneous iterates at every level. □

Example 6.7. For $F(x_1\mathbf{e}_1+x_2\mathbf{e}_2)=(0.4x_1+a_1)\mathbf{e}_1+(0.8x_2+a_2)\mathbf{e}_2$ on $\mathbb{D}^m$ with the minimum multi-norm, $L=0.4\mathbf{e}_1+0.8\mathbf{e}_2$ and $x^*=(a_1/0.6)\mathbf{e}_1+(a_2/0.2)\mathbf{e}_2$.

Proof. The affine difference eliminates a_j and scales the components by 0.4 and 0.8. The minimum multi-norm obeys the same component bound for every tuple. Solving $x_j=Ljx_j+a_j$ gives the stated fixed point. □

Proposition 6.8. Operator pullbacks preserve comparisons. If J:E→F is injective and $c\|x\|\mathbb{D}\preccurlyeq\mathbb{D}\|Jx\|\mathbb{D}\preccurlyeq\mathbb{D}C\|x\|\mathbb{D}$ with positive invertible c,C, then the pullback of the minimum multi-norm satisfies

$cN_n^{min,E}\preccurlyeq\mathbb{D}N_n^J\preccurlyeq\mathbb{D}CN_n$^{min,E}. (6.4)

Proof. Apply the two-sided estimate to every coordinate and take component maxima. This gives a quantitative version of the pullback construction and shows that a stable embedding produces a fixed-level and uniform multi-norm equivalence in the minimum case. □

The examples suggest three different roles. The minimum multi-norm measures worst-case amplitude; the Hilbert multi-norm measures how a family can be assembled through orthogonal channels; and pullback multi-norms measure families after a prescribed observation map. In uncertain systems, the two idempotent coefficients may carry separate stability margins. In optimization, an iterate family can be controlled without averaging away its largest error. In learning models, an operator-generated multi-norm can be tied to a feature map, but selecting that map and validating its relevance remain substantive modelling tasks.

6.9 Order Intervals and Robust Enclosures

Definition 6.9. For $a,b\in\mathbb{D}$ with $a\preccurlyeq\mathbb{D}b$, the hyperbolic order interval is $[a,b]\mathbb{D}=\{z:a\preccurlyeq\mathbb{D}z\preccurlyeq\mathbb{D}b\}$. A family error enclosure is $N_n(e_1,\ldots,e_n)\preccurlyeq\mathbb{D}\varepsilon$, with $\varepsilon=\varepsilon_1\mathbf{e}_1+\varepsilon_2\mathbf{e}_2$.

Proposition 6.10. If T is multi-bounded and $N_n(e)\preccurlyeq\mathbb{D}\varepsilon$, then $N_n(Te_1,\ldots,Te_n)\preccurlyeq\mathbb{D}\|T\|mb,\mathbb{D}\varepsilon$. If T is componentwise invertible and bounded below by m with positive coefficients, the reverse enclosure $N_n(e)\preccurlyeq\mathbb{D}m^{-1}N_n(Te)$ holds for minimum multi-norms.

Proof. The forward estimate is the definition of the multi-bound. In the minimum case, the component lower bounds $\|T_jx_j\|\geq m_j\|x_j\|$ apply to each coordinate, so taking maxima yields $mN_n(e)\preccurlyeq\mathbb{D}N_n(Te)$. Multiplying by the invertible positive m^{-1} gives the reverse estimate. □

Example 6.11. Suppose a sensor map has gains between 0.9 and 1.1 in component one and between 0.5 and 1.5 in component two. An observed family error δ therefore implies a state-error enclosure with coefficients $\delta_1/0.9$ and $\delta_2/0.5$, not with a single reciprocal of an averaged gain.

Proof. The lower-bound constant is $m=0.9\mathbf{e}_1+0.5\mathbf{e}_2$. Its inverse is $(1/0.9)\mathbf{e}_1+2\mathbf{e}_2$. Applying Proposition 6.10 gives the component enclosure. A scalar average could understate the second-channel uncertainty. □

Order intervals make the intended interpretation of hyperbolic values explicit: they are paired enclosures, not complex amplitudes. This is useful in robust optimization and interval-like modelling, but only when the two channels have a justified semantic meaning. Multi-norms then extend an enclosure from one state to arbitrary finite families without changing the tolerance convention.

7. Applications and Interpretive Analysis

In the field of stability theory, a family estimate $N_n(F(x_1)-G(x_1),\ldots,F(x_n)-G(x_n))\preccurlyeq\mathbb{D}\varepsilon$ is capable of simultaneously controlling a finite number of deviations. The worst-case control is given by the minimum multi-norm, whereas Hilbert multi-norms can register an orthogonal distribution among components. Separate tolerances are maintained for the two idempotent channels in hyperbolic values.

Two scenarios, signed modes, or independently calibrated submodels may be represented by a state $x=x_1e_1+x_2e_2$ for uncertain systems.. $L=L_1e_1+L_2e_2$ is a contraction constant that indicates the stability of both scenarios. Fixed-point conclusions necessitate $L_1<1$ and $L_2<1$; the product order stipulates that a single contractive component cannot compensate for an expansive component.

Batch iterates can be evaluated in optimisation using $N_n(x_1-x,\ldots,x_n-x)$. A total order for competing objective components is not determined by the theory; therefore, Pareto or scalarization rules are still required. Bicomplex and hypercomplex neural models have been algebraically investigated in the field of machine learning (Alpay et al., 2023). A decomposable hyperbolic multi-norm can quantify families of activations or perturbations in paired channels; however, empirical robustness or predictive gains necessitate separate experiments.

Function-space examples can uniformly control trajectories over time, and operator-generated constructions can encode a measurement system Γ. These roles are mathematically credible due to the explicit norming and finiteness conditions of each construction.

8. Open Problems and Limitations

The majority of the examples in this paper are decomposable. The interaction may be modelled by a coupled construction whose first coefficient is contingent upon both idempotent components; however, it must maintain the partial order and every cross-level axiom. This location does not presently offer a general classification. In high dimensions, the Hilbert multi-norm may be computationally demanding due to its supremum over projection decompositions.

Some of the open problems include explicit maximum hyperbolic multi-norms, uniform equivalence constants, tensor and operator-space versions, duality, compactness, interpolation between minimum and Hilbert structures, and stochastic estimates. Algorithms that compare component performance without silently replacing the partial order by a scalar norm are necessary for applications to split-complex optimisation and learning. Additionally, these algorithms must respect null-cone values.

9. In summary

An extensive yet rigorous example theory is present in the hyperbolic multi-norm framework. Each necessary axiom is satisfied by the minimum, Hilbert-projection, function-space, sequence-space, pullback, and norming-operator constructions through componentwise verification. Sum and Euclidean tuple gauges, despite being legitimate norms at each level, are not repeatable and must not be relabelled. The relationship between the constructions is clarified by comparison inequalities and finite-dimensional fixed-level equivalence, while null-cone examples illustrate the unnecessariness and danger of hyperbolic division. The examples provide support for precise stability and operator interpretations and identify specific questions for future analytical and computational research.

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